NUMERICAL VERIFICATION FOR EXISTENCE OF A GLOBAL-IN-TIME SOLUTION TO SEMILINEAR PARABOLIC EQUATIONS

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Abstract. This paper presents a method of numerical verification for existence of a global-in-time solution to a class of semilinear parabolic equations. Such a method is based on two main theorems in this paper. One theorem gives a sufficient condition for proving existence of a solution to the semilinear parabolic equations with the initial point $t = t' \geq 0$. If the sufficient condition does not hold, the other theorem is used for enclosing the solution for time $t \in (0, \tau]$, $\tau > 0$ in a neighborhood of a numerical solution. Numerical results of obtaining a global-in-time solution for a certain semilinear parabolic equation are also given.

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Date: March 31, 2015 (Submitted); December 10, 2015 (Revised).
2010 Mathematics Subject Classification. Primary 65G40, 65M15; Secondary 35K20.
Key words and phrases. semilinear parabolic equations, global-in-time solution, verified numerical computations.
1. Introduction

Let $\Omega$ be a bounded and convex domain in $\mathbb{R}^2$. We consider existence of a global-in-time solution for the following semilinear parabolic equations:

\begin{align}
\begin{cases}
\partial_t u - \Delta u = f(x, u) & \text{in } (0, \infty) \times \Omega, \\
u(t, x) = 0 & \text{on } (0, \infty) \times \partial \Omega, \\
u(0, x) = u_0(x) & \text{in } \Omega,
\end{cases}
\end{align}

where $\partial_t u = \frac{\partial u}{\partial t}$, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian, the domain of the Laplacian is $\mathcal{D}(\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$, $u_0 \in H^1_0(\Omega)$ is an initial function, and $f$ is a function from $\Omega \times H^1_0(\Omega)$ to $L^2(\Omega)$ in the sense that $f(\cdot, u) \in L^2(\Omega)$ for each $u \in H^1_0(\Omega)$. The map $f : H^1_0(\Omega) \rightarrow L^2(\Omega)$ defined in this sense is the twice Fréchet differentiable. The main aim of this paper is to present Theorem 3.1 in Section 3.1 and Theorem 3.2 in Section 3.2. Then we propose an algorithm for numerically verifying existence of a global-in-time solution to (1).

Existence of global-in-time solutions for some parabolic equations related to (1) has been shown by using analytic approaches. For example, for the parabolic equation (1a) and (1c) when $f(x, u) = u^p \left( \frac{2}{m} - \frac{2}{p} \right)$ and $\Omega = \mathbb{R}^m$, H. Fujita has found an exponent concerning existence of a global-in-time solution in 1966 [1]. From the pioneering work by H. Fujita, studies of solutions to various parabolic equations have been developed in the field of mathematical analysis ([2, 3, 4, 5], etc). For the parabolic equation (1), existence of the global-in-time solution that converges to the zero function has been shown and the following estimate has been obtained (c.f. Theorem 19.2 in [6]): there exist $\nu > 0$ and $\tilde{\rho} > 0$ such that

\begin{equation}
\|u(t)\|_{L^\infty} \leq \tilde{\rho} e^{-\nu t}, \quad t > 0
\end{equation}

holds, where $\tilde{\rho} > 0$ depends on $\|u_0\|_{L^\infty}$.

The main progress of this paper compared with the previous analytical results is to explicitly give the values $\nu > 0$ and $\tilde{\rho} > 0$ corresponding to (2). In this paper presenting these values is called quantification of the analytical result. In order to obtain these values, this paper also presents a verification algorithm. The algorithm tries to enclose a global-in-time solution that exponentially converges to a stationary solution of (1) by numerically checking whether sufficient conditions in Theorem 3.1 and Theorem 3.2 hold, respectively.

M.T. Nakao, T. Kinoshita, and T. Kimura have proposed a novel computer-assisted method for enclosing a solution to a class of parabolic equations based on verified numerical computations [7, 8, 9]. Their method is based on estimating a norm of an inverse operator for the parabolic equations. Moreover S. Cai has derived a sufficient condition that is related to existence of a solution for time $t > t'$, $t' \geq 0$ to a system of reaction-diffusion equations by using an analytic semigroup over $L^\infty(\Omega) \times L^\infty(\Omega)$ in [10].

Recently we have developed a method for verifying existence of a solution to a semilinear parabolic equation by using an analytic semigroup over $H^{-1}(\Omega)$ (a topological dual space of $H^1_0(\Omega)$) in [11]. In this paper by using an analytic semigroup over $L^2(\Omega)$, we consider a mild solution of (1), whose definition is given in

\footnote{A solution that exists for $t \in (0, \infty)$ is called a global-in-time solution. We consider the solution of (1) in $L^\infty((0, \infty); H^1_0(\Omega))$ in this paper.}
Section 2. The main progress of this paper from the previous paper is a residual estimate. Namely the residual estimate in $L^2(\Omega)$ is tighter than the residual estimate in $H^{-1}(\Omega)$. Comparison of both estimates is given in Appendix A. We will show a method for verifying existence of a global-in-time solution. In such a method, existence of a global-in-time solution for (1) is shown by the following procedure: First we check whether the sufficient condition in Theorem 3.1 holds. If this condition holds, we can show existence of a global-in-time solution. Otherwise we try to enclose a mild solution $u(t)$ for $t \in (0, \tau]$, $\tau > 0$ in a neighborhood of a numerical solution to (1) by checking whether (13) denoted in Theorem 3.2 holds. If the enclosure of the solution is obtained, we also verify existence of the mild solution for (1) is guaranteed in a subset of the Banach space $L^\infty (\Omega)$. Comparison of both estimates is given in Appendix A. We also define a function space $H^1_0(\Omega)$ as $\{ u \in H^1(\Omega) : u = 0$ on $\partial \Omega \}$. For a positive integer $j$, let $H^j(\Omega)$ be the $j$th order Sobolev space of $L^2(\Omega)$. We define a function space $H^j_0(\Omega)$ such that $\| u \|_{H^j_0} := \| \nabla^j u \|_{L^2}$. For any interval $J$, let $L^\infty(J) := \{ u : \text{ess sup}_t \in J | u(t) | < \infty \}$. Let a function space $Y$ be a Banach space with the norm $\| \cdot \|_Y$. We define a function space $L^\infty(J; Y)$ as

$$ L^\infty(J; Y) := \left\{ u : J \times \Omega \rightarrow \mathbb{R}, u(t, \cdot) \in Y : \text{ess sup}_{t \in J} \| u(t, \cdot) \|_Y < \infty \right\} $$

with the norm $\| \cdot \|_{L^\infty(J; Y)} := \text{ess sup}_{t \in J} \| u(t, \cdot) \|_Y$. Let $C^0(J)$ be the function space of all continuous functions from $J$ to $\mathbb{R}$. We also define a function space $C^0(J; Y)$ as

$$ C^0(J; Y) := \left\{ u : J \times \Omega \rightarrow \mathbb{R} \mid u(t, \cdot) \in Y, \| u(t, \cdot) \|_Y \in C^0(J) \right\}. $$

Let $P$ and $Q$ be Banach spaces. For a bounded operator $B : P \rightarrow Q$, the operator norm of $B$ is denoted by $\| B \|_{P,Q}$.
We denote $A = -\Delta : D(A) \to L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. We define $\rho(A)$ as a resolvent set of $A$:

$$\rho(A) := \{ z \in \mathbb{C} \mid (zI - A)^{-1} : L^2(\Omega) \to L^2(\Omega) \text{ exists and is a bounded operator} \}.$$ 

Let $\sigma(A) = \mathbb{C} \setminus \rho(A)$, $\lambda_{\min}$ denotes the minimum value of $\sigma(A)$. For $0 \leq \alpha \leq 1$, a fractional operator of $A$ is defined by

$$A^\alpha u := \sum_{j=1}^{\infty} \lambda_j^\alpha c_j \psi_j, \quad D(A^\alpha) := \left\{ u = \sum_{j=1}^{\infty} c_j \psi_j \in L^2(\Omega) : \sum_{j=1}^{\infty} c_j^2 \lambda_j^{2\alpha} < \infty \right\},$$

where $\{\psi_j\}_{j \in \mathbb{N}}$ is a complete orthogonal basis of eigenfunctions in $L^2(\Omega)$, $c_j = \langle u, \psi_j \rangle_{L^2}$, and $\{\lambda_j\}_{j \in \mathbb{N}} = \sigma(A)$.

Let $\{e^{-tA}\}_{t \geq 0}$ be an analytic semigroup generated$^2$ by $-A$.

**Definition 2.1.** Let $J = [t_0, t_1]$ ($0 \leq t_0 < t_1 \leq \infty$). For the semilinear parabolic equation:

$$\begin{align*}
\partial_t u - \Delta u &= f(x, u) \quad \text{in } J \times \Omega, \\
u(t, x) &= 0 \quad \text{on } J \times \partial \Omega, \\
u(t_0, x) &= \nu_0(x) \quad \text{in } \Omega,
\end{align*}$$

(3)

the function $u \in C^0(J; L^2(\Omega))$ given by

$$u(t) = e^{-(t-t_0)A}u_0 + \int_{t_0}^{t} e^{-(t-s)A}f(s, \nu(s))ds \quad (t \in J)$$

is a mild solution of (3) on $J$.

Then we introduce Lemma 2.2 and Lemma 2.3 that are given in [12, 13].

**Proposition 2.2.** $D(A^{1/2}) = H^1_0(\Omega)$ and

$$\|u\|_{H^1_0} = \|A^{1/2}u\|_{L^2}, \forall u \in H^1_0(\Omega)$$

hold.

**Proposition 2.3.** Let $\alpha \in (0, 1]$. If $u \in D(A^\alpha)$, then

$$A^\alpha e^{-tA}u = e^{-tA}A^\alpha u, \quad t > 0$$

holds.

Furthermore, we obtain the following lemma:

**Proposition 2.4.** Let $\lambda_{\min}$ be the minimum eigenvalue of $A$. For fixed $\alpha \in (0, 1)$ and $\beta \in (0, 1)$, the following estimate holds:

$$\|A^\alpha e^{-tA}\|_{L^2, L^2} \leq \left(\frac{\alpha}{\epsilon \beta}\right)^\alpha t^{-\alpha} e^{-(1-\beta)t\lambda_{\min}}, \quad t > 0.$$

**Proof.** Since the minimum eigenvalue of $A$ is positive, we have

$$\sup_{x \in (\lambda_{\min}, \infty)} |x^\alpha e^{-\beta t x}| \leq \left(\frac{\alpha}{\epsilon \beta}\right)^\alpha \frac{\epsilon}{2\beta} \quad \text{and} \quad \sup_{x \in (\lambda_{\min}, \infty)} |e^{-(1-\beta)t x}| \leq e^{-(1-\beta)t\lambda_{\min}}$$

$^2$It is known [12, 13] that $-A$ generates an analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ over $L^2(\Omega)$. 
for fixed $\alpha \in (0, 1)$ and $\beta \in (0, 1)$. From the spectrum mapping theorem the following inequality holds:
\begin{equation}
\|A_0 e^{-tA}\|_{L^2, L^2} = \sup_{x \in (\lambda_{\min}, \infty)} |x^\alpha e^{-tx}| \\
(6)
\leq \sup_{x \in (\lambda_{\min}, \infty)} |x^\alpha e^{-\beta tx}| \sup_{x \in (\lambda_{\min}, \infty)} |e^{-1-\beta tx}|.
\end{equation}
This indicates that the inequality (5) holds. \hfill \Box

For $x > 0$, the error function $\text{erf}(x)$ is defined by
\begin{equation}
\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds.
\end{equation}

By an elemental calculation it follows for $\alpha > 0$ and $x > 0$,
\begin{equation}
\int_0^x s^{-1/2} e^{-as} ds = \frac{2}{\sqrt{a}} \text{erf}(\sqrt{ax}).
\end{equation}

Let $\rho > 0$ and $J$ be any interval. For $v \in L^\infty(J; H^1_0(\Omega))$, a closed ball $B_J(v, \rho)$ is defined by
\begin{equation}
B_J(v, \rho) := \left\{ y \in L^\infty(J; H^1_0(\Omega)) : \|y - v\|_{L^\infty(J; H^1_0(\Omega))} \leq \rho \right\}.
\end{equation}

For $y \in B_J(v, \rho)$ and $s \in J$, $J^\prime[y(s)] : H^1_0(\Omega) \rightarrow L^2(\Omega)$ denotes a Fréchet derivative of $f$ at $y(s)$. Let $(f(\cdot, u), v)_{L^2} = \int_{\Omega} f(x, u(x))v(x) dx$ for $u \in H^1_0(\Omega)$ and $v \in L^2(\Omega)$.

We also denote $\|f(\cdot, u)\|_{L^2} = \sqrt{\int_{\Omega} |f(x, u(x))|^2 dx}$ for $u \in H^1_0(\Omega)$.

3. Numerical verification for a global-in-time solution

3.1. Global-in-time existence theorem. Let $\phi \in \mathcal{D}(A)$ be a stationary solution of \eqref{eq:1}. Namely $\phi$ satisfies
\begin{equation}
\left\{ \begin{array}{l}
A\phi = f(x, \phi) \quad \text{in} \ \Omega, \\
\phi = 0 \quad \text{on} \ \partial\Omega.
\end{array} \right.
\end{equation}

A function space $V_h$ denotes a finite dimensional subspace\(^3\) of $\mathcal{D}(A)$ depending on a parameter $h > 0$. We assume that the stationary solution $\phi$ is uniquely enclosed\(^4\) in the ball:
\begin{equation}
B_{H^1_0}(\phi, \rho') := \left\{ \mu \in H^1_0(\Omega) : \|\mu - \phi\|_{H^1_0} \leq \rho' \right\} \quad \text{for} \ \rho' > 0,
\end{equation}
where $\phi \in V_h$ is a certain numerical solution of $\phi$.

In this subsection we give an inequality that presents a sufficient condition of enclosing a mild solution $u(t)$ of \eqref{eq:1} with the initial point $t = 0$ replaced by some $t = t' > 0$ in a neighborhood of the stationary solution $\phi$. In the following we consider a mild solution of \eqref{eq:1} for $t \in (t', \infty)$ satisfying
\begin{equation}
\left\{ \begin{array}{l}
\partial_t u + Au = f(x, u) \quad \text{in} \ (t', \infty) \times \Omega, \\
u(t, x) = 0 \quad \text{on} \ (t', \infty) \times \partial\Omega, \\
u(t', x) = \eta \quad \text{in} \ \Omega,
\end{array} \right.
\end{equation}

\(^3\)For example, $V_h$ is a $C^1$-finite element subspace. Alternatively, when $\Omega$ is a rectangular domain, $V_h$ is spanned by the Fourier bases.
\(^4\)One can easily check whether a stationary solution $\phi$ uniquely exists in $B_{H^1_0}(\phi, \rho')$ by using various computer-assisted methods, e.g \cite{14, 15, 16}.
then a mild solution (13) highly depends on the analytical result corresponding to (2). Some examples are also given in Section 4.

**Theorem 3.1.** We consider the semilinear parabolic equation (11). We assume that a stationary solution $\phi \in C(\Omega)$ is uniquely exists in $B_{H_0^1}(\phi, \rho')$ defined by (10). We also assume that there exists a non-decreasing function $L_\phi : \mathbb{R} \rightarrow \mathbb{R}$ such that a stationary solution $\phi$ is defined by

$$L_\phi(1) \in \text{support of } \phi.$$  

We also assume that there exists a non-decreasing function $L_\phi$ that depends on $\phi$. Let $\lambda$ satisfy $0 \leq \lambda < \lambda_{\min}/2$. If there exists $\rho > 0$ such that

$$\|\eta - \phi\|_{H_0^1} + L_\phi(\rho)\frac{2\pi}{e(\lambda_{\min} - 2\lambda)} < \rho,$$

then a mild solution $u(t)$ of (11) uniquely exists in

$$B_{X_\lambda}(\phi, \rho) := \{ u \in L^\infty((t', \infty); H_0^1(\Omega)) : \|u - \phi\|_{X_\lambda} \leq \rho \}.$$  

Therefore the following estimate holds:

$$\|u(t) - \phi\|_{H_0^1} \leq \rho e^{-(t-t')\lambda}, \quad t \in (t', \infty).$$

**Remark.** A non-decreasing function $L_\phi$ is essential for our verification method because existence of $\rho > 0$ satisfying (13) highly depends on the $L_\phi$. For example there exists $L_\phi$ given in (12) if $f$ is a polynomial, i.e. $f(x, u) = \sum_{i=1}^N c_i u^i$, where $N \in \mathbb{N}$ and $c_i \in \mathbb{R}$. However such a non-decreasing function $L_\phi$ does not exist if $f(x, u) = u^{1/2}$.

**Remark.** Since $\|\eta - \tilde{u}\|_{H_0^1} \leq \epsilon$ and a stationary solution $\phi$ exists in $B_{H_0^1}(\tilde{\phi}, \rho')$, it follows

$$\|\eta - \phi\|_{H_0^1} \leq \|\eta - \tilde{u}\|_{H_0^1} + \|\tilde{u} - \tilde{\phi}\|_{H_0^1} + \|\tilde{\phi} - \phi\|_{H_0^1},$$

where we remark that $\|\tilde{u} - \tilde{\phi}\|_{H_0^1}$ can be rigorously computable by using interval arithmetic (e.g. INTLAB [17]). Therefore $\|\eta - \phi\|_{H_0^1}$ in Theorem 3.1 can be estimated rigorously.

**Proof of Theorem 3.1.** A nonlinear operator $S : L^\infty((t', \infty); H_0^1(\Omega)) \rightarrow L^\infty((t', \infty); H_0^1(\Omega))$ is defined by

$$(Sz)(t) := e^{-(t-t')A}(\eta - \phi) + \int_{t'}^t e^{-(t-s)A} (f(\cdot, z(s) + \phi) - f(\cdot, \phi)) \, ds, \quad t \in (t', \infty).$$
We note the solution \( u(t) \) of (11) is a mild solution if and only if \( S \) has a fixed point \( z \). Let \( Z := \{z \in X_\lambda : \|z\|_{X_\lambda} < \rho \} \) for a certain \( \rho > 0 \). We derive a condition based on Banach's fixed-point theorem so that \( S \) has a fixed-point in \( Z \).

Let \( z \in Z \). For \( s \in (t', \infty) \) and \( 0 \leq \theta \leq 1 \), put a real-valued function \( k(\theta) := (f(\cdot, \phi + \theta z(s)), v)_{L^2Z} \), \( \forall v \in L^2(\Omega) \). From the mean-value theorem for all \( v \in L^2(\Omega) \), there exists \( \xi_v \in (0, 1) \) such that

\[
k'(\xi_v) = k(1) - k(0) = (f(\cdot, \phi + z(s)) - f(\cdot, \phi), v)_{L^2Z},
\]

where \( k' \) is the first-derivative of \( k \) corresponding to \( \theta \). From \( k'(\xi_v) := (f'[\phi + \xi_v z(s)]z(s), v)_{L^2Z} \) for \( s \in (t', \infty) \),

\[
(f(\cdot, \phi + z(s)) - f(\cdot, \phi), v)_{L^2Z} = (f'(\xi_v z(s))z(s), v)_{L^2Z}
\]

holds. For \( s \in (t', \infty) \), the Schwarz inequality gives

\[
|\langle (e^{(s-t')\Lambda}(f(\cdot, \phi + z(s)) - f(\cdot, \phi)), v)_{L^2Z} \rangle| = |\langle (e^{(s-t')\Lambda})(f(\cdot, \phi + \xi_v z(s))z(s), v)_{L^2Z} \rangle|
\]

\[
= |\langle f'(\xi_v z(s))z(s), v)_{L^2Z} \rangle| \leq \| (f'[\phi + \xi_v z(s)]e^{(s-t')\Lambda}z(s), v)_{L^2Z} \| \|v\|_{L^2Z}.
\]

Since \( \xi_v \in (0, 1) \) holds for all \( v \in L^2(\Omega) \), \( y_v := \phi + \xi_v z \in B_{X_\lambda}(\phi, \rho) \subset B_{(\nu, \infty)}(\phi, \rho) \). Substituting \( y_v \) into the function \( y \) in (12), we have

\[
|\langle (e^{(s-t')\Lambda}(f(\cdot, \phi + z(s)) - f(\cdot, \phi)), v)_{L^2Z} \rangle| \leq \text{ess sup}_{s \in (t', \infty)} \| (f'(y_v(s))e^{(s-t')\Lambda}z(s), v)_{L^2Z} \| \|v\|_{L^2Z}
\]

\[
\leq L_y(\rho)\|z\|_{X_\lambda}\|v\|_{L^2Z}. \tag{14}
\]

From the Riesz representation theorem and (14)

\[
\| (e^{(s-t')\Lambda}(f(\cdot, \phi + z(s)) - f(\cdot, \phi)), v)_{L^2Z} \| \leq \sup_{v \in L^2(\Omega) \setminus \{0\}} \frac{|\langle (e^{(s-t')\Lambda}(f(\cdot, \phi + z(s)) - f(\cdot, \phi)), v)_{L^2Z} \rangle|}{\|v\|_{L^2Z}} \leq L_y(\rho)\|z\|_{X_\lambda} \tag{15}
\]

holds. Then (4) yields

\[
e^{(t-s')\Lambda}(S_{\lambda}(t))\|H_1^s\|
\]

\[
= e^{(t-s')\Lambda}\|A^{1/2}(S_{\lambda}(t))\|_{L^2Z}
\]

\[
\leq e^{(t-s')\Lambda}\|A^{1/2}e^{-(t-s')A}(\eta - \phi)\|_{L^2Z}
\]

\[
+ e^{(t-s')\Lambda}\int_s^t\|A^{1/2}e^{-(t-s')A}\|_{L^2Z_L}\|f(\cdot, z(s) + \phi) - f(\cdot, \phi)\|_{L^2Z}ds
\]

\[
= e^{(t-s')\Lambda}\|A^{1/2}e^{-(t-s')A}(\eta - \phi)\|_{L^2Z}
\]

\[
+ \int_s^t e^{(t-s')\Lambda}\|A^{1/2}e^{-(t-s')A}\|_{L^2Z_L}e^{(s-t')\Lambda}\|f(\cdot, z(s) + \phi) - f(\cdot, \phi)\|_{L^2Z}ds. \tag{16}
\]
Since \( \lambda < \lambda_{\text{min}}/2 \) and \( \int_{t'}^t (t-s)^{-1/2} e^{-(t-s)(\lambda_{\text{min}}-\lambda)/2} ds < \infty \) for a fixed \( t > t' \) hold, (16) and Lemma 2.4 imply
\[
e^{(t-t')\lambda} \| (S z)(t) \|_{H_0^1}
\leq e^{(t-t')\lambda} A^{1/2} e^{-(t-t')A}(\eta - \phi) \|_{L^2}
+ \text{ess sup}_{s \in (t', \infty)} \left( e^{(s-t')\lambda} \| f(\cdot, z(s) + \phi) - f(\cdot, \phi) \|_{L^2} \right) e^{-1/2} \int_{t'}^t (t-s)^{-1/2} e^{-(t-s)\frac{\lambda_{\text{min}}-\lambda}{2}} ds.
\]
Moreover from (15) and the spectrum mapping theorem
\[
e^{(t-t')\lambda} \| (S z)(t) \|_{H_0^1}
\leq e^{(t-t')\lambda (\lambda - \lambda_{\text{min}})} \| \eta - \phi \|_{H_0^1}
+ e^{-1/2} L_\phi(\rho) \| z \|_{X_s} \int_{t'}^t (t-s)^{-1/2} e^{-(t-s)\frac{\lambda_{\text{min}}-\lambda}{2}} ds
\]
holds. It follows from (7), (17), and \( \lambda < \lambda_{\text{min}}/2 \)
\[
e^{(t-t')\lambda} \| (S z)(t) \|_{H_0^1} \leq \| \eta - \phi \|_{H_0^1} + L_\phi(\rho) \| z \|_{X_s} \sqrt{\frac{2\pi}{e^{(\lambda_{\text{min}}-2\lambda)(t-t')}}}
\]
Since \( \text{erf}(x) \) is a monotonically increasing function for \( x > 0 \) and \( \text{erf}(x) \to 1 \) as \( x \to \infty \), we have
\[
\| S z(t) \|_{X_s} \leq \| \eta - \phi \|_{H_0^1} + L_\phi(\rho) \| z \|_{X_s} \sqrt{\frac{2\pi}{e^{(\lambda_{\text{min}}-2\lambda)}}}.
\]
Therefore if \( \rho > 0 \) satisfies (13), \( S(z) \in Z \) holds.
For \( \lambda < \lambda_{\text{min}}/2 \) it follows from \( \psi_i \in Z \) (\( i = 1, 2 \))
\[
e^{(t-t')\lambda} \| (S \psi_1)(t) - (S \psi_2)(t) \|_{H_0^1}
\leq \int_{t'}^t e^{(t-s)\lambda} A^{1/2} e^{-(t-s)A} L^2 \| (f(\cdot, \psi_1(s) + \phi) - f(\cdot, \psi_2(s) + \phi)) \|_{L^2} ds
\leq \text{ess sup}_{s \in (t', \infty)} \left( e^{(s-t')\lambda} \| f(\cdot, \psi_1(s) + \phi) - f(\cdot, \psi_2(s) + \phi) \|_{L^2} \right)
\times e^{-1/2} \int_{t'}^t (t-s)^{-1/2} e^{-(t-s)\frac{\lambda_{\text{min}}-\lambda}{2}} ds
\leq L_\phi(\rho) \| \psi_1 - \psi_2 \|_{X_s} e^{-1/2} \int_{t'}^t (t-s)^{-1/2} e^{-(t-s)\frac{\lambda_{\text{min}}-\lambda}{2}} ds.
\]
From (7) we obtain
\[
e^{(t-t')\lambda} \| (S \psi_1)(t) - (S \psi_2)(t) \|_{H_0^1} \leq L_\phi(\rho) \sqrt{\frac{2\pi}{e^{(\lambda_{\text{min}}-2\lambda)(t-t')}}} \| \psi_1 - \psi_2 \|_{X_s}.
\]
Then it sees that
\[
\| S(\psi_1) - S(\psi_2) \|_{X_s} \leq L_\phi(\rho) \sqrt{\frac{2\pi}{e^{(\lambda_{\text{min}}-2\lambda)}}} \| \psi_1 - \psi_2 \|_{X_s}.
\]
If \( \rho > 0 \) satisfies (13), \( L_\phi(\rho)\sqrt{\frac{2\pi}{e(\lambda_{\min} - 2\lambda)}} < 1 \) holds. Then \( S \) becomes a contraction mapping on \( Z \). Banach’s fixed-point theorem proves that a fixed point of \( S \) uniquely exists in \( Z \).

In order to verify existence of a global-in-time solution to (1) we set \( t' = 0 \) in (11). Then we check whether the sufficient condition in Theorem 3.1 holds. If this condition holds, we can show existence of the global-in-time solution in \( L^\infty((0, \infty); H^1_0(\Omega)) \). Otherwise we try to enclose a mild solution of (1) for \( t \in (0, \tau], \tau > 0 \) in a neighborhood of a numerical solution. Such a procedure is introduced in the next subsection.

### 3.2. Verification algorithm

For fixed \( t_0 \) and \( t_1 \) satisfying \( 0 \leq t_0 < t_1 < \infty \) let \( J := (t_0, t_1] \) and \( \tau := t_1 - t_0 \). In this subsection we give a sufficient condition for guaranteeing existence and local-in-time uniqueness (Theorem 3.2) of a mild solution to (1) for time \( t \in J \). We also give an a posteriori error estimate in Corollary 3.3.

Let \( \hat{u}_k \in V_h \) \( (k = 0, 1) \). Then we consider (1) for \( t \in J = (t_0, t_1] \), i.e.

\[
\begin{aligned}
\eta \frac{\partial u + Au = f(x, u)}{u(t, x) = \xi} & \text{ in } J \times \Omega, \\
\eta u(t, x) = 0 & \text{ on } J \times \partial \Omega, \\
\eta u(t, x) = \xi & \text{ in } \Omega,
\end{aligned}
\]

where \( \xi \in H^1_0(\Omega) \) satisfies \( \|\xi - \hat{u}_0\|_{H^1_0} \leq \varepsilon \) for \( \varepsilon > 0 \). Let \( \phi_k(t) \) \( (t \in J) \) be a linear Lagrange basis satisfying \( \phi_k(t_j) = \delta_{k_j} \) \( (j = 0, 1) \), where \( \delta_{k_j} \) is Kronecker’s delta. We define \( \omega_0(t) \) as

\[
\omega_0(t) = \hat{u}_0 \phi_0(t) + \hat{u}_1 \phi_1(t), \quad t \in J.
\]

In the following we give a sufficient condition for guaranteeing existence and local uniqueness of a mild solution in \( B_J(\omega_0, \rho) \) for a certain \( \rho > 0 \).

**Theorem 3.2.** We consider the semilinear parabolic equation (20). Let \( \delta \) denote

\[
\delta = \left\| \int_{t_0}^t A^{1/2} e^{-(t-s)A} (\eta \omega_0(s) + A \omega_0(s) - f(\cdot, \omega_0(s))) ds \right\|_{L^\infty(J; L^2(\Omega))},
\]

where \( \omega_0 \) is defined by (21). We assume that there exists a non-decreasing function \( L_{\omega_0} : \mathbb{R} \to \mathbb{R} \) such that for \( y \in B_J(\omega_0, \rho) \)

\[
\|\eta^t y\|_{L^\infty(J; L^2(\Omega))} \leq L_{\omega_0}(\rho) \|y\|_{L^\infty(J; H^1_0(\Omega))}, \quad \forall u \in L^\infty(J; H^1_0(\Omega)),
\]

where the function \( L_{\omega_0} \) depends on \( \omega_0 \).

If \( \rho > 0 \) satisfies

\[
\varepsilon + \frac{2\pi}{\lambda_{\min}} \text{erf} \left( \sqrt{\frac{\lambda_{\min} \tau}{2}} \right) L_{\omega_0}(\rho) \rho + \delta < \rho,
\]

then a mild solution \( u(t) \) of (20) for \( t \in J \) uniquely exists in \( B_J(\omega_0, \rho) \).

**Proof.** By using the analytic semigroup \( e^{-tA} \), an operator \( S : L^\infty(J; H^1_0(\Omega)) \to L^\infty(J; H^1_0(\Omega)) \) is defined by

\[
(Sz)(t) := e^{-(t-t_0)A}(\xi - \hat{u}_0) + \int_{t_0}^t e^{-(t-s)A} g(z(s)) ds,
\]
where we put \( g(z(t)) := f(x, z(t) + \omega_0(t)) - \partial_t \omega_0(t) - A \omega_0(t) \). We note that the solution \( u(t) \) of (20) is a mild solution if and only if \( S \) has a fixed point \( z \). We derive a condition based on Banach’s fixed-point theorem so that \( S \) has a fixed-point in \( B_J(0, \rho) \) for a certain \( \rho > 0 \).

At first we derive a condition guaranteeing \( S(B_J(0, \rho)) \subset B_J(0, \rho) \). By using (4), Lemma 2.3, and the spectrum mapping theorem, the first term in the right-hand side of (25) is estimated by

\[
\left\| e^{-(t-t_0)A}(\xi - \hat{u}_0) \right\|_{H^1_0} = \left\| e^{-(t-t_0)A}A^{1/2}(\xi - \hat{u}_0) \right\|_{L^2} \leq e^{-(t-t_0)\lambda_{\text{min}}} \varepsilon.
\]  

(26)

Then we have

\[
\left\| e^{-(t-t_0)A}(\xi - \hat{u}_0) \right\|_{L^\infty(J,H^1_0(\Omega))} \leq \varepsilon.
\]  

(27)

Next we divide \( g(z(\cdot)) \in L^2(\Omega) \) in (25) into the following two parts:

\[
g_1(s) := f(x, z(s) + \omega_0(s)) - f(x, \omega_0(s)) \quad \text{and} \quad g_2(s) := f(x, \omega_0(s)) - \partial_t \omega_0(s) - A \omega_0(s).
\]

From (4) and Lemma 2.4 we estimate

\[
\left\| \int_{t_0}^{t} A^{1/2} e^{-(t-s)A} g_1(s) ds \right\|_{L^2} \leq \int_{t_0}^{t} \left\| A^{1/2} e^{-(t-s)A} g_1(s) \right\|_{L^2} ds \leq e^{-1/2 \nu(t)} \| g_1 \|_{L^\infty(J,L^2(\Omega))},
\]  

(28)

where \( \nu(t) \) is denoted by

\[
\nu(t) := \int_{t_0}^{t} (t-s)^{-1/2} e^{-1/2(t-s)\lambda_{\text{min}}} ds.
\]

(29)

From (7) the supremum of \( \nu(t) \) for \( t \in J \) is derived by

\[
\sup_{t \in J} \nu(t) = \sqrt{\frac{2\pi}{\lambda_{\text{min}}} \text{erf} \left( \sqrt{\frac{\lambda_{\text{min}}}{2}} \right)}.
\]

For \( s \in J, 0 \leq \theta \leq 1 \), put a real-valued function \( k(\theta) := (f(\cdot, \omega_0(s) + \theta z(s)), v)_{L^2} \), \( \forall v \in L^2(\Omega) \). From the mean-value theorem for \( v \in L^2(\Omega) \), there exists \( \xi_v \in (0, 1) \) such that

\[
k'(\xi_v) = k(1) - k(0) = (f(\cdot, \omega_0(s) + z(s)) - f(\cdot, \omega_0(s)), v)_{L^2},
\]

where \( k' \) is the first derivative corresponding to \( \theta \). From \( k'(\xi_v) := (f'[\omega_0(s) + \xi_v z(s)]z(s), v)_{L^2} \), \( s \in J \),

\[
(f(\cdot, \omega_0(s) + z(s)) - f(\cdot, \omega_0(s)), v)_{L^2} = (f'[\omega_0(s) + \xi_v z(s)]z(s), v)_{L^2}
\]

holds. For \( s \in J \), the Schwarz inequality gives

\[
|((f(\cdot, \omega_0(s) + z(s)) - f(\cdot, \omega_0(s))), v)_{L^2}| = |(f'[\omega_0(s) + \xi_v z(s)]z(s), v)_{L^2}|

= |(f'[\omega_0(s) + \xi_v z(s)]z(s), v)_{L^2}|

\leq \|(f'[\omega_0(s) + \xi_v z(s)]z(s))\|_{L^2} \|v\|_{L^2}.
\]
Since \( \xi_v \in (0, 1) \) holds for all \( v \in L^2(\Omega) \), \( y_v := \omega_0 + \xi_v z \in B_f(\omega_0, \rho) \). Substituting \( y_v \) into the function \( y \) in (23) it follows
\[
\| (f(\cdot, \omega_0(s) + z(s)) - f(\cdot, \omega_0(s)), v)_{L^2} \| \leq \sup_{s \in J} \| (f'(y_v(s)) z(s)) \|_{L^2} \| v \|_{L^2}
\]
From (27), (32) and (22) we have
\[
\| f(\cdot, \omega_0(s) + z(s)) - f(\cdot, \omega_0(s)) \|_{L^2} = \sup_{v \in L^2(\Omega) \setminus \{0\}} \frac{| (f(\cdot, \omega_0(s) + z(s)) - f(\cdot, \omega_0(s)), v)_{L^2} |}{\| v \|_{L^2}}
\]
holds. Therefore (28), (29) and (31) give
\[
\left\| \int_{t_0}^t A^{1/2} e^{-(t-s)A} g(t) \right\|_{L^\infty(J; L^2(\Omega))} \leq \sqrt{\frac{2\pi}{\lambda_{\min} e}} \operatorname{erf} \left( \sqrt{\frac{\lambda_{\min} \tau}{2}} \right) L_{\omega_0}(\rho) \rho.
\]
From (27), (32) and (22) we have
\[
\| S(z) \|_{L^\infty(J; L^2(\Omega))} \leq \varepsilon + \sqrt{\frac{2\pi}{\lambda_{\min} e}} \operatorname{erf} \left( \sqrt{\frac{\lambda_{\min} \tau}{2}} \right) L_{\omega_0}(\rho) \rho + \delta.
\]
The condition (24) yields that \( \| S(z) \|_{L^\infty(J; L^2(\Omega))} < \rho \) holds. Namely, \( S(z) \in B_f(0, \rho) \) is obtained.

Furthermore we obtain a sufficient condition under which \( S \) becomes a contraction mapping on \( B_f(0, \rho) \). Let \( z_1, z_2 \in B_f(0, \rho) \). From the definition of \( S \) it follows
\[
(Sz_1)(t) - (Sz_2)(t) = \int_{t_0}^t e^{-(t-s)A} \left\{ f(\cdot, z_1(s) + \omega_0(s)) - f(\cdot, z_2(s) + \omega_0(s)) \right\} ds.
\]
Since \( z_i + \omega_0 \in B_f(\omega_0, \rho) \) \((i = 1, 2)\), we have the following estimate from (4), (23), (29), and Lemma 2.4:
\[
\| (Sz_1) - (Sz_2) \|_{L^\infty(J; H^1_0(\Omega))} \leq \sqrt{\frac{2\pi}{\lambda_{\min} e}} \operatorname{erf} \left( \sqrt{\frac{\lambda_{\min} \tau}{2}} \right) L_{\omega_0}(\rho) \| z_1 - z_2 \|_{L^\infty(J; H^1_0(\Omega))}.
\]
The condition (24) implies
\[
\sqrt{\frac{2\pi}{\lambda_{\min} e}} \operatorname{erf} \left( \sqrt{\frac{\lambda_{\min} \tau}{2}} \right) L_{\omega_0}(\rho) < 1.
\]
Then \( S \) becomes a contraction mapping on \( B_f(0, \rho) \).

Finally Banach's fixed-point theorem yields that a fixed-point uniquely exists in \( B_f(0, \rho) \). Such a fixed-point \( z(t) \) for \( t \in J \) can be expressed by
\[
z(t) = e^{-(t-t_0)A}(\xi - \tilde{u}_0) + \int_{t_0}^t e^{-(t-s)A} g(z(s)) ds.
\]

Moreover we obtain the following a posteriori error estimate at \( t = t_1 \) if Theorem 3.2 holds.
Corollary 3.3. Under the same assumption in Theorem 3.2, $\delta$ denotes
\begin{equation}
\delta = \left\| \int_{t_0}^{t_1} A^{1/2} e^{-(t-s)A}(\partial_x \omega_0(s) + A\omega_0(s) - f(\cdot, \omega_0(s))) ds \right\|_{L^2(\Omega)}.
\end{equation}

Then the mild solution $u(t_1)$ of (20) satisfies
\begin{equation}
\|u(t_1) - \tilde{u}_1\|_{H_0^1} \leq e^{-\tau \lambda_{\min} \varepsilon} + \sqrt{\frac{2\pi}{\lambda_{\min} \varepsilon}} \text{erf} \left( \sqrt{\frac{\lambda_{\min} \tau}{2}} \right) L_{\omega_0}(\rho) \rho + \delta.
\end{equation}

Proof. Let $u(t_1)$ be the mild solution of (20) at $t = t_1$. From (34) $u(t_1) - \tilde{u}_1$ is described by
\begin{equation}
u(t_1) - \tilde{u}_1 = e^{-(t_1-t_0)A}(\xi - \tilde{u}_0) + \int_{t_0}^{t_1} e^{-(t_1-s)A} g(z(s)) ds,
\end{equation}
where $g(z(s)) = f(x, z(s) + \omega_0(s)) - A\omega_0(s) - \partial_x \omega_0(s)$. By using (33) in Theorem 3.2
\begin{equation}
\|u(t_1) - \tilde{u}_1\|_{H_0^1} \leq e^{-\tau \lambda_{\min} \varepsilon} + \sqrt{\frac{2\pi}{\lambda_{\min} \varepsilon}} \text{erf} \left( \sqrt{\frac{\lambda_{\min} \tau}{2}} \right) L_{\omega_0}(\rho) \rho + \delta
\end{equation}
holds. \hfill \Box

By using Theorem 3.2 we explain how to numerically enclose the solution of (1) for $t \in (0, r]$, $\tau > 0$. We set $t_0 = 0$ and $t_1 = \tau$. Then we try to find $\rho > 0$ satisfying the inequality (24). If the $\rho > 0$ is obtained, the mild solution for $t \in (0, \tau]$ is enclosed in $B_{(0, \tau]}(\omega_0, \rho)$ on the basis of Theorem 3.2.

Let us set $\varepsilon > 0$ as the right-hand side of (36). We substitute $\varepsilon$ to (13) and check existence of the mild solution $u(t)$ of (1) for $t \in (\tau, \infty)$ by using Theorem 3.1. By repeatedly using Theorem 3.1, Theorem 3.2, and Corollary 3.3 we try to obtain a global-in-time solution to (1). On the basis of Theorem 3.1, Theorem 3.2, and Corollary 3.3, we introduce a verification algorithm for existence of a global-in-time solution. We give the verification algorithm in Algorithm 1.

In Algorithm 1, each ball $C_{T_k}$ ($k = 1, 2, \cdots, n$) is an enclosure of the solution to (1) for $t \in T_k$. Let us define $C_T$ as
\begin{equation}
C_T := \{ y \in L^\infty(T; H_0^1(\Omega)) : y(t) \in C_{T_k}, \ t \in T_k, \ k = 1, 2, \ldots, n \}.
\end{equation}
If Algorithm 1 finishes successfully, we can show that a solution $u(t)$ of (1) for $t \in T$ is enclosed in $C_T$. Moreover the solution is asymptotic to $\phi$ for $t \in (t', \infty)$. Therefore existence of a global-in-time solution to (1) can be proved by verified numerical computations.

Remark. If the global-in-time solution $u(t)$ is enclosed by Algorithm 1, the solution $u(t) \in H_0^1(\Omega) \subset L^2(\Omega) \subset L^2(\Omega)$ for $t \in [0, \infty)$ is expressed by
\begin{equation}
u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(\cdot, u(s)) ds.
\end{equation}
The solution $u$ is in $C^0([0, \infty); L^2(\Omega))$. Details of the proof is given in Appendix C.
Algorithm 1 Verification algorithm

Set \( \hat{\phi} \in V_h \);
Verify existence and local uniqueness of a stationary solution \( \phi \) in \( B_{H^1_0}(\hat{\phi}, \rho') \)
defined by (10);
if Failed in enclosing \( \phi \) then
  error ("Failed in enclosing \( \phi \"));
end if
Set \( \hat{u}_0 \in V_h \) and compute \( \varepsilon > 0 \) satisfying \( \| u_0 - \hat{u}_0 \|_{H^1_0} \leq \varepsilon \);
\( t' = 0; \eta = u_0; \hat{u} = \hat{u}_0; k = 0; \)
while true do
  Compute \( \| \hat{\eta} - \hat{\phi} \|_{H^1_0} \) based on Remark 3.1;
  Choose \( \lambda \) satisfying \( 0 \leq \lambda < \lambda_{\min}/2 \);
  if There exists \( \rho > 0 \) satisfying (13) in Theorem 3.1 then
    break;
  end if
  \( k = k + 1; \)
  \( u_0 = \hat{u}; t_0 = t'; \xi = \eta; \)
  Set \( \tau > 0. \) Let \( t_1 = t_0 + \tau \) and \( T_k = (t_0, t_1] \);
  Choose \( \hat{u}_1 \in V_h \) and set \( \omega_0(t) = \hat{u}_0 \phi_0(t) + \hat{u}_1 \phi_1(t) \) for \( t \in T_k \);
  Compute \( \delta \) defined by (22);
  if there exists \( \rho > 0 \) satisfying (24) in Theorem 3.2 then
    Define a ball \( C_{T_k} \) as \( B_{T_k}(\omega_0, \rho) \) and \( \rho_k = \rho; \)
    Compute \( \delta \) defined by (35);
    Substituting \( \rho \) for the right-hand side of (36), compute \( \varepsilon > 0 \) satisfying
    \( \| u(t_1) - \hat{u}_1 \|_{H^1_0} \leq \varepsilon \) based on Corollary 3.3;
  else
    error ("Verification failed for \( t \in T_k. "");
  end if
  \( t' = t_1; \eta = u(t_1); \hat{u} = \hat{u}_1; \)
end while
\( n = k; \)
\( \text{disp} ("The solution for } t \in (0, t'] \text{ exists and } \| u(t) - \phi \|_{H^1_0} \leq \rho e^{-\lambda(t-t')} \text{ holds for } t > t' ");\)

4. Numerical results

Let \( \phi \in H^1_0(\Omega) \) be a stationary solution of (1). In (2), we replace \( u \) by \( u - \phi \).
Then using Algorithm 1, we show existence of the global-in-time solution for (1) by giving values of the following constants \( \lambda > 0 \) and \( \rho > 0 \) such that
\[
\| u(t) - \phi \|_{H^1_0} \leq \rho e^{-\lambda(t-t')} , \quad t > t' \geq 0. \tag{39}
\]
This section gives values \( \lambda > 0, \rho > 0, \) and \( t' \geq 0 \) satisfying (39).
Let \( \Omega := \{(x, y) : 0 < x, y < 1\} \subset \mathbb{R}^2 \) be an unit square domain. We consider existence of global-in-time solutions for the following semilinear parabolic equations:
\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u &= f_i(x, y, u) \quad \text{in } (0, \infty) \times \Omega, \\
u(t, x) &= 0 \quad \text{on } (0, \infty) \times \partial \Omega, \\
u(0, x) &= 2\sin(\pi x)\sin(\pi y) \quad \text{in } \Omega,
\end{align*}
\tag{40}
\]
where \( f_i(x, y, u) \) \((i = 1, 2, 3, 4)\) is set by
\[
\begin{align*}
  f_1(x, y, u) &= u^2 + 4\sin(\pi x)\sin(\pi y), \\
  f_2(x, y, u) &= u^2 + 4(\sin(\pi x)\sin(\pi y) + \sin(2\pi x)\sin(2\pi y) + \sin(\pi x)\sin(2\pi y)), \\
  f_3(x, y, u) &= u^2 + 4 \sum_{1 \leq k, l \leq 2} \sin(k\pi x)\sin(l\pi y), \\
  f_4(x, y, u) &= u^2 + 4 \sum_{1 \leq k, l \leq 3} \sin(k\pi x)\sin(l\pi y).
\end{align*}
\]

All computations are carried out on CentOS 6.3 with 3.10GHz Intel(R) Xeon(R) CPU E5-2687W, 128GB RAM. We use MATLAB 2012b with INTLAB ver.7.1 [17]. The spectrum method is employed for discretizing the spatial variable. Namely, we construct a numerical solution by using the Fourier bases. For \( N \), a finite dimensional subspace \( V_N \subset \mathcal{D}(A) \) is defined by
\[
V_N := \left\{ u \in \mathcal{D}(A) : u(x, y) = \sum_{k, l=1}^N a_{k, l} \sin(k\pi x)\sin(l\pi y), a_{k, l} \in \mathbb{R} \right\}.
\]

We fix \( N = 10 \). We set \( \tau = 2^{-8} \) and \( \lambda = 1/40 \) in Algorithm 1. Then we try to verify existence of global-in-time solutions to (40) by using Algorithm 1.

Let \( \psi_i \) \((i = 1, 2, 3, 4)\) denote stationary solutions of (40). We verify existence and local uniqueness of \( \psi_i \) in a neighborhood of a numerical solution \( \tilde{\psi}_i \in \mathcal{V}_h \) by using the verification method given in [16]. A radius of the neighborhood is denoted by \( \rho_i \) \((i = 1, 2, 3, 4)\) satisfying \( \|\psi_i - \tilde{\psi}_i\|_{H^1} \leq \rho_i \). Each \( \rho_i \) is shown in Table 1. The numerical solutions \( \tilde{\psi}_i \) are displayed in Figure 1.

**Table 1.** Radii of the neighborhood enclosing \( \psi_i \) \((i = 1, 2, 3, 4)\) when \( N = 10 \).

<table>
<thead>
<tr>
<th>i</th>
<th>( \rho_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.002706328809</td>
</tr>
<tr>
<td>2</td>
<td>0.003861742749</td>
</tr>
<tr>
<td>3</td>
<td>0.004967902695</td>
</tr>
<tr>
<td>4</td>
<td>0.00724564522</td>
</tr>
</tbody>
</table>

For simplicity, in the following we consider (40) when \( i = 1 \). Let \( \hat{u}_0 \in V_N \) be a numerical solution of (40) at time \( t = t_0 \). We give a numerical solution \( \hat{u}_1 \in V_N \) of (40) at time \( t = t_1 \) in Algorithm 1 as follows. We employ the Crank-Nicolson scheme in order to get each \( \hat{u}_1 \in V_N \), i.e. we consider the following problem: for \( \hat{u}_0 \in V_N \), find \( u_1 \in V_N \) such that
\[
\left( \frac{u_1 - \hat{u}_0}{\tau}, v_N \right)_{L^2} + \frac{1}{2} (A\hat{u}_0 + Au_1, v_N)_{L^2} = \frac{1}{2} \left( f_1(\cdot, \cdot, \hat{u}_0) + f_1(\cdot, \cdot, u_1), v_N \right)_{L^2},
\]
where for \( w = \hat{u}_0 \) or \( u_1 \), \( f_1(\cdot, \cdot, w), v_N)_{L^2} = \int_0^1 \int_0^1 f_1(x, y, w(x, y))v_N(x, y)dxdy \). Let \( \hat{u}_1 \in V_N \) be a numerical solution of \( u_1 \). We define a numerical solution \( \omega_0 \) as
\[
\omega_0(t) = \hat{u}_0\phi_0(t) + \hat{u}_1\phi_1(t), \quad t \in T_k
\]
The numerical solution for existence of a global-in-time solution

Figure 1. The numerical solutions \( \hat{\psi}_i \) for \( i = 1, 2, 3, \) and 4.

in Algorithm 1. We compute \( \delta \) in (22), \( \tilde{\delta} \) in (35), \( L_\omega(\rho) \) in (12), and \( L_{\omega^\infty}(\rho) \) in (23) for (40) based on each estimate in Appendix. Then Algorithm 1 gives \( \rho_k > 0 \) satisfying

\[
\|u - \omega_0\|_{L^\infty(T_k; H_1^0(\Omega))} \leq \rho_k.
\]

Figure 2a displays each \( \rho_k \) for \( T_k \) when \( N = 10 \) and \( \tau = 2^{-8} \).

When \( i = 2, 3, \) and 4 in (40), Figure 2 also shows each \( \rho_k \) for \( T_k \) when \( N = 10 \) and \( \tau = 2^{-8} \). Furthermore the algorithm 1 gives the following estimates:

\[
\|u(t) - \psi_i\|_{H_1^0} \leq \rho_k e^{-(t-t_i')/40}, \quad t \in (t_i', \infty) \quad (i = 1, 2, 3, 4).
\]

Table 2 also shows each error estimate \( \rho_i \) and \( t_i' \) of (43).

Table 2. Error estimate \( \rho_i \) and \( t_i' \) are presented when \( N = 10 \) and \( \tau = 2^{-8} \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \rho_i )</th>
<th>( t_i' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.973712650429328</td>
<td>0.1015625</td>
</tr>
<tr>
<td>2</td>
<td>0.939460907598910</td>
<td>0.10546875</td>
</tr>
<tr>
<td>3</td>
<td>0.953394626139478</td>
<td>0.10546875</td>
</tr>
<tr>
<td>4</td>
<td>0.954276545574080</td>
<td>0.11328125</td>
</tr>
</tbody>
</table>

On the other hand, when we consider (40) for \( i = 1 \) and \( u(0, x) = 5.5 \sin(\pi x) \sin(\pi y) \), Algorithm 1 fails in enclosing a global-in-time solution because existence of the solution \( u(t) \) for \( t > 0.16796875 \) cannot be shown. Figure 3 displays each \( \rho_k \) for \( T_k \) when \( N = 10 \) and \( \tau = 2^{-8} \). In Algorithm 1, as repeatedly using Theorem 3.2 and
Corollary 3.3, the error $\varepsilon$ in (24) tends to accumulate, so called wrapping effect. When we see Figure 3, the accumulated error seems to prevent from enclosing a global-in-time solution of (40). Therefore Algorithm 1 cannot verify existence of a global-in-time solution of (40) for this example.

**Appendix A. Residual estimation**

We derive estimation of $\delta$ in (22) and $\tilde{\delta}$ in (35).

For fixed $t_0$ and $t_1$ such that $0 \leq t_0 < t_1 < \infty$, let $J = (t_0, t_1]$ and $\tau = t_1 - t_0$. $V_N$ is the same as that in Section 4. We remind that $f$ is defined by (1). For $u_0^N \in V_N$, we employ the Crank-Nicolson scheme in order to get $\tilde{u}_1 \in V_N$, i.e. for $u_0^N \in V_N$, we will find $u_1^N \in V_N$ such that

$$
(u_1^N - u_0^N, v_N)_{L^2} + \frac{1}{2} \left( A(u_0^N + u_1^N), v_N \right)_{L^2} = \frac{1}{2} (f(\cdot, u_0^N) + f(\cdot, u_1^N), v_N)_{L^2}
$$

(44)
for any $v_N \in V_N$. For $j = 0, 1$, let $\tilde{u}_j \in V_N$ be a numerical solution of $u_j^N \in V_N$, respectively. Let $\phi_k (k = 0, 1)$ is a linear Lagrange basis satisfying $\phi_k(t_j) = \delta_{k,j}$ ($k, j = 0, 1$), where $\delta_{k,j}$ is Kronecker’s delta. Then we define $\omega_0 \in L^\infty(J;V_N)$ as

$$\omega_0(t) = \tilde{u}_0\phi_0(t) + \tilde{u}_1\phi_1(t), \quad t \in J.$$ 

For a fixed $\theta$ satisfying $0 \leq \theta \leq 1$, we define $C_\theta \in L^2(\Omega)$ as

$$C_\theta := \frac{\tilde{u}_1 - \tilde{u}_0}{\tau} + (1 - \theta)A\tilde{u}_0 + \theta A\tilde{u}_1 - (1 - \theta)f(\cdot, \tilde{u}_0) - \theta f(\cdot, \tilde{u}_1).$$

Let $\Phi(t) := f(\cdot, \tilde{u}_1)\phi_1(t) + f(\cdot, \tilde{u}_0)\phi_0(t)$ for $t \in J$. We consider the following two parts:

$$\begin{align*}
\| \int_{t_0}^t A^{1/2}e^{-(t-s)A} & \left( f(\cdot, \omega_0(s)) - \partial_s \omega_0(s) - A\omega_0(s) \right) ds \|_{L^2} \\
\leq & \left\|\int_{t_0}^t A^{1/2}e^{-(t-s)A} \left( f(\cdot, \omega_0(s)) - \Phi(s) \right) ds \right\|_{L^2} \\
& + \left\|\int_{t_0}^t A^{1/2}e^{-(t-s)A} \left( \Phi(s) - \partial_s \omega_0(s) - A\omega_0(s) \right) ds \right\|_{L^2}.
\end{align*}$$

(45)

We estimate the first term of (45). Since both $\tilde{u}_0$ and $\tilde{u}_1$ are in $V_N \subset L^\infty(\Omega)^5$, a classical error bound of linear interpolation yields for fixed $x \in \Omega$,

$$|f(x, \omega_0(t)) - \Phi(t)| \leq \frac{\tau^2}{8} \max_{t \in J} \left| \frac{d^2}{dt^2} f(x, \omega_0(t)) \right|$$

$$= \frac{\tau^2}{8} \max_{t \in J} \left| f''(\omega_0(t)) \left( \frac{d\omega_0}{dt} \right)^2 \right|$$

$$= \frac{1}{8} \max_{t \in J} \left| f''(\omega_0(t)) \right| \left| (\tilde{u}_1 - \tilde{u}_0)^2 \right|.$$
From (53), it follows

\[(46) \quad \|f'(\cdot, \omega_0(t)) - \Phi(t)\|_{L^2} \leq \frac{C_2^2}{8} |f'| [\omega_0]\|_{L^\infty(J; L^\infty(\Omega))} \|\tilde{u}_1 - \tilde{u}_0\|_{H^1_0}^2.\]

From (46),

\[\int_{t_0}^t \|A^{1/2} e^{-(t-s)A}(f'(\cdot, \omega_0(s)) - \Phi(s))\|_{L^2} ds \leq e^{-1/2} \int_{t_0}^t (t - s)^{-1/2} e^{-(t-s)\lambda_{\min}} \|f'(\cdot, \omega_0(s)) - \Phi(s)\|_{L^2} ds\]

\[\leq \sqrt{\frac{2\pi}{\lambda_{\min}}} \text{erf} \left( \sqrt{\frac{\lambda_{\min}(t - t_0)}{2}} \right) \|f'(\cdot, \omega_0) - \Phi\|_{L^\infty(J; L^2(\Omega))}\]

holds. Therefore we obtain the following upper bound:

\[(47) \quad \left\|\int_{t_0}^t A^{1/2} e^{-(t-s)A}(f'(\cdot, \omega_0(s)) - \Phi(s)) ds\right\|_{L^\infty(J; L^2(\Omega))} \leq C_\rho \alpha^2 \sqrt{\frac{2\pi}{\lambda_{\min}}} \text{erf} \left( \sqrt{\frac{\lambda_{\min}(t - t_0)}{2}} \right),\]

where we put

\[C_\rho := \frac{C_2^2}{8} |f'| [\omega_0]\|_{L^\infty(J; L^\infty(\Omega))} \text{ and } \alpha := \|\tilde{u}_1 - \tilde{u}_0\|_{H^1_0}.\]

We estimate the second term of (45). Since \(\phi_1(s) + \phi_0(s) = 1 (s \in J)\) holds, we have

\[\Phi(s) - \partial_s \omega_0(s) - A\omega_0(s) = -(C_1 \phi_1(s) + C_0 \phi_0(s)) = -(C_1 - C_0) \phi_1(s) + (C_0 - C_0) \phi_0(s) + C_\rho = -(C_0 + C_1 - C_0)(1 - \theta) \phi_1(s) - \theta \phi_0(s)).\]

Then for a fixed \(t \in J\), it sees that

\[\left\|\int_{t_0}^t A^{1/2} e^{-(t-s)A}(\Phi(s) - \partial_s \omega_0, \theta(s) - A\omega_0, \theta(s)) ds\right\|_{L^2} \leq \int_{t_0}^t e^{-1/2} \|C_\rho\|_{L^2} (t - s)^{-1/2} e^{-(t-s)\lambda_{\min}} \frac{2}{2} ds\]

\[\leq \int_{t_0}^t e^{-1/2} \|C_\rho\|_{L^2} (t - s)^{-1/2} e^{-(t-s)\lambda_{\min}} \frac{2}{2} ds\]

\[\leq \sqrt{\frac{2\pi}{\lambda_{\min}}} \text{erf} \left( \sqrt{\frac{\lambda_{\min}(t - t_0)}{2}} \right) \left(\|C_\rho\|_{L^2} + \max(\theta, 1 - \theta)\|C_1 - C_0\|_{L^1}\right).\]

Therefore when \(\theta = 1/2\), both \(\delta\) and \(\tilde{\delta}\) are bounded by

\[\sqrt{\frac{2\pi}{\lambda_{\min}}} \text{erf} \left( \sqrt{\frac{\lambda_{\min}(t - t_0)}{2}} \right) \left(C_\rho \alpha^2 + \|C_2\|_{L^2} + \frac{\|C_1 - C_0\|_{L^2}}{2}\right).\]

Here we sketch a difference between this paper and [11]. In [11] we give a sufficient condition for enclosing a solution to (20) by using an analytic semigroup over \(H^{-1}(\Omega)\), where \(H^{-1}(\Omega)\) is the topological dual space of \(H^1_0(\Omega)\). Let \(\langle \cdot, \cdot \rangle\) be a
dual product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$. A linear operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined by

$$\langle Au, v \rangle := (\nabla u, \nabla v)_{L^2}, \quad \forall v \in H_0^1(\Omega).$$

The operator $-A$ generates an analytic semigroup \{e^{-tA}\}_{t \geq 0}. We define $\delta_{-1}$ as

$$\delta_{-1} = \left\| \int_0^t e^{-(t-s)A}(\partial_t \omega_0(s) + A\omega_0(s) - f(\cdot, \omega_0(s))) ds \right\|_{L^\infty(J, H_0^1(\Omega))}.$$

The sufficient condition for enclosing a solution of (20) given in [11] is that there exists $\rho > 0$ satisfying

$$\varepsilon + 2\sqrt{\frac{\tau}{e}}L_{\omega_0}(\rho) + \delta_{-1} < \rho,$$

where we recall that $\varepsilon$ and $L_{\omega_0}$ are given in Theorem 3.2. The main difference of (24) and (51) is $\delta_{-1}$. To estimate $\delta_{-1}$, let us define two functionals $B(\hat{u}_1) \in H^{-1}(\Omega)$ and $F(\hat{u}_1) \in H^{-1}(\Omega)$ as

$$\langle B(\hat{u}_1), v \rangle := \left( \frac{\hat{u}_1 - \hat{u}_0}{\tau}, v \right)_{L^2} + (\nabla \hat{u}_1, \nabla v)_{L^2} - (f(\cdot, \hat{u}_1), v)_{L^2}, \forall v \in H_0^1(\Omega),$$

$$\langle F(\hat{u}_1), v \rangle := \left( \frac{\hat{u}_1 - \hat{u}_0}{\tau}, v \right)_{L^2} + (\nabla \hat{u}_1, \nabla v)_{L^2} - (f(\cdot, \hat{u}_0), v)_{L^2}, \forall v \in H_0^1(\Omega),$$

respectively. We obtain

$$\delta_{-1} \leq \frac{1}{4} \sqrt{\frac{[\Omega] \tau}{e}} \| f''(\omega_0) \|_{L^\infty(J, L^\infty(\Omega))} \| \hat{u}_1 - \hat{u}_0 \|_{L^\infty} + \left\{ \beta + \left( \frac{1 - e^{\tau \lambda_{\min}}}{\tau \lambda_{\min}} \right) (\beta + \eta) \right\},$$

where $[\Omega]$ is the measure of $\Omega$, $\beta = \| B(\hat{u}_1) \|_{H^{-1}}$, and $\eta = \| F(\hat{u}_1) \|_{H^{-1}}$. Here we note that both $\beta$ and $\eta$ can be estimated rigorously by using methods given in [14], [15], and [16].

We numerically compare $\delta$ with $\delta_{-1}$. We consider (40) when $i = 1$ with the interval $(0, \infty)$ replaced by $(0, 2^{-8}]$. We set a numerical solution $\omega_0$ as (41). Then we estimate $\delta$ and $\delta_{-1}$ given in (50) and (52), respectively. The values of $\delta$ and $\delta_{-1}$ are given in Table 3. Table 3 shows an advantage of the numerical verification.

<table>
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<th>Method</th>
<th>$\delta$</th>
<th>$\delta_{-1}$</th>
</tr>
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<td>NUMERICAL VERIFICATION FOR EXISTENCE OF A GLOBAL-IN-TIME SOLUTION</td>
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<td>0.34954941203</td>
</tr>
</tbody>
</table>

We derive $L_{\omega}(\rho)$ in (12) and $L_{\omega_0}(\rho)$ in (23) when $f(x, u) = u^2 + g$, where $g \in L^2(\Omega)$ is a given function. Let $g$ be a natural number. There exists Sobolev’s embedding constant $C_{e, g} > 0$ [18] satisfying

$$\| u \|_{L^q} \leq C_{e, g} \| u \|_{H^1}, \quad \forall u \in H_0^1(\Omega),$$

where $\| \cdot \|_{L^q}$ represents the norm in the usual Lebesgue $L^q(\Omega)$ space. Such a constant $C_{e, g}$ can be estimated (see Lemma 2 in [15] for example). Let $J$ be any
interval. For \( \rho > 0 \) and a given \( v \in L^\infty(J; H_0^1(\Omega)) \), let \( w \in B_J(v, \rho) \). Here for \( u \in L^\infty(J; H_0^1(\Omega)) \) and a fixed \( s \in J \), we can obtain
\[
\|f'(w(s))u(s)\|_{L^2} = 2\|w(s)u(s)\|_{L^2} \\
\leq 2\|w(s)\|_{L^4}\|u(s)\|_{L^4} \\
\leq 2C_{s,A}^2\|w(s)\|_{H^1_0}\|u\|_{L^\infty(J; H_0^1(\Omega))} \\
\leq 2C_{s,A}^2(\rho + \|v\|_{L^\infty(J; H_0^1(\Omega))})\|u\|_{L^\infty(J; H_0^1(\Omega))}.
\]
Therefore \( L_\phi(\rho) = 2C_{s,A}^2(\rho + \|\phi\|_{L^\infty(J; H_0^1(\Omega))}) \) and \( L_\omega(\rho) = 2C_{s,A}^2(\rho + \|\omega_0\|_{L^\infty(J; H_0^1(\Omega))}) \) holds, respectively. Furthermore we estimate
\[
\|\phi\|_{L^\infty(T_\kappa; H_0^1(\Omega))} \leq \rho' + \|\hat{\phi}\|_{H_0^1} \\
\|\omega_0\|_{L^\infty(T_\kappa; H_0^1(\Omega))} \leq \max \left\{ \|\tilde{\omega}_0\|_{H_0^1}, \|\tilde{\phi}_k\|_{H_0^1} \right\},
\]
respectively.

**Appendix C. The continuity of the global-in-time solution**

If existence of the global-in-time solution to (1) is proved by Algorithm 1, the solution \( u(t) \in H_0^1(\Omega) \subset L^2(\Omega) \) for \( t \in [0, \infty) \) is expressed by
\[
u(t) = e^{-tA}u_0 + \int_0^te^{-(t-s)A}f(\cdot, u(s))ds.
\]
We will show the solution \( u \) is in \( C^\theta([0, \infty); L^2(\Omega)) \).

First, we will show \( f(\cdot, u) \in L^\infty((0, \infty); L^2(\Omega)) \). Recall that \( t' \geq 0, \rho > 0 \), the stationary solution \( \phi \in H_0^1(\Omega), n \in N, T_k (1 \leq k \leq n) \), and \( \rho_k (1 \leq k \leq n) \) are defined in Algorithm 1. For \( \omega_k \in C^1(T_k; D(A)) \) and \( z_k \in B_{T_k}(\omega_k, \rho_k) \) the solution is expressed by \( u = z_k + \omega_k \). Moreover for \( 1 \leq k \leq n \), there exists \( \delta_k < \infty \) such that \( \|\omega_k - \phi\|_{L^\infty(J; H_0^1(\Omega))} \leq \delta_k \). From (31) for \( s \in T_k \),
\[
\|f(\cdot, u(s))\|_{L^2} \leq \|f(\cdot, z_k(s) + \omega_k(s)) - f(\cdot, \omega_k(s))\|_{L^2} + \|f(\cdot, \omega_k(s)) - f(\cdot, \phi)\|_{L^2} \\
+ \|f(\cdot, \phi)\|_{L^2} \\
\leq L_{\omega_k}(\rho_k)\rho_k + L_{\omega_k}(\delta_k)\delta_k + \|f(\cdot, \phi)\|_{L^2} < \infty
\]
holds. From (15), it follows that for \( s \in (t', \infty) \)
\[
\|f(\cdot, u(s))\|_{L^2} \leq \|f(\cdot, u(s)) - f(\cdot, \phi)\|_{L^2} + \|f(\cdot, \phi)\|_{L^2} \\
\leq L_\phi(\rho) + \|f(\cdot, \phi)\|_{L^2} < \infty.
\]
Therefore, there exists \( M < \infty \) such that \( \|f(\cdot, u)\|_{L^\infty((0, \infty); L^2(\Omega))} \leq M \) and the constant \( M \) is independent of both the space variable and the time variable.
Next, we will show the global-in-time solution \( u \) is in \( C^0([0, \infty); L^2(\Omega)) \). Fix \( t' \geq 0 \). For \( t \in \mathbb{R} \) such that \( 0 \leq t' < t < \infty \), we have

\[
\| u(t) - u(t') \|_{L^2} \leq \left\| (e^{-tA} - e^{-t'A})u_0 \right\|_{L^2} + \left\| \int_0^t e^{-(t-s)A} f(\cdot, u(s))ds - \int_0^{t'} e^{-(t'-s)A} f(\cdot, u(s))ds \right\|_{L^2} \\
\leq \| (e^{-tA} - e^{-t'A})u_0 \|_{L^2} + \int_0^t \| e^{-(t-s)A} \|_{L^2,L^2} \| f(\cdot, u(s)) \|_{L^2} ds + \int_{t'}^t \| e^{-(t'-s)A} \|_{L^2,L^2} \| f(\cdot, u(s)) \|_{L^2} ds \\
\leq \| (e^{-tA} - e^{-t'A}) \|_{L^2,L^2} \| u_0 \|_{L^2} + \frac{M(1 - e^{-\lambda_{\min}(t-t')})}{\lambda_{\min}} + \| e^{-(t'-t)A} - I \|_{L^2,L^2} M(1 - e^{-\lambda_{\min}t'})
\]

(54)

where \( I \) is an identity operator from \( L^2(\Omega) \) to \( L^2(\Omega) \). Note that we use the spectral mapping theorem and Hölder’s inequality in the last term of (54). From the continuity of the semigroup, \( \| e^{-tA} - e^{-t'A} \|_{L^2,L^2} \to 0 \) and \( \| e^{-(t'-t)A} - I \|_{L^2,L^2} \to 0 \) (e.g. [12]) hold, respectively if \( t \to t' + 0 \). Then the right hand side of (54) tends to 0 if \( t \to t' + 0 \). On the contrary, we fix \( t' > 0 \). For \( t \in \mathbb{R} \) such that \( 0 < t < t' < \infty \), we estimate \( \| u(t) - u(t') \|_{L^2} \) by the same way as (54) after exchanging \( t \) with \( t' \) in (54).

By using the continuity of the semigroup, \( \| u(t') - u(t) \|_{L^2} \) tends to 0 if \( t \to t' - 0 \). Therefore, the global-in-time solution \( u \) is in \( C^0([0, \infty); L^2(\Omega)) \).

Acknowledgements

The Authors express their sincere thanks to Prof. Masahide Kashiwagi and Dr. Takuma Kimura in Waseda University for their useful comments. The authors also would like to express sincere thanks to anonymous referees for giving us useful comments to improve this paper. This work was supported in part by Grant for Basic Science Research Projects from The Sumitomo Foundation.

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